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Qualitatively stable finite difference schemes for advection–reaction equations

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Abstract

A systematic procedure is proposed and implemented for the design of nonstandard finite difference methods as reliable numerical simulations that preserve significant properties inherent to the solutions of advection–reaction equations. In the case of hyperbolic fixed-points, a renormalization of the denominators of the discrete derivatives is performed for the numerical solutions to display the linear stability properties of the exact solutions. Non-hyperbolic fixed-points are described with the help of two new monotonic properties the construction of schemes, which preserve these properties, being done by nonlocal approximation of nonlinear terms in the reaction terms.

MSC: 65M06; 65M99

Keywords: Advection–reaction equations; Nonstandard finite difference method; Stability with respect to a property

1. Introduction

The general framework of this paper is the following initial-value problem for the advection–reaction equation ($a \geqslant 0$ and $b \geqslant 0$)

$$\begin{cases} \partial_t u + a\partial_x u + b\partial_y u = r(u), \\ u(x,y,0) = f(x,y), \end{cases} \tag{1}$$

that arises, for instance, in acoustics and fluid dynamics. We assume once and for all that (1) has a unique solution. In what follows, it is implicitly understood that the reaction r and the function f satisfy the needed differentiability properties.

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We shall use the mesh of discrete points $\{t_k := k\Delta t\}_{k \geqslant 0}$, $\{x_m := m\Delta x\}_{m\in\mathbb{Z}}$ and $\{y_n := n\Delta y\}_{n\in\mathbb{Z}}$ for the t, x and y variables, respectively. We denote by $u^k_{m,n}$ an approximation to the solution u at the point (t_k, x_m, y_n): $u^k_{m,n} \approx u(t_k, x_m, y_n)$. Our main task is to design, for (1), finite difference schemes of the form

$$\partial_{\Delta t} u^k_{m,n} + a\partial_{\Delta x} u^k_{m,n} + b\partial_{\Delta y} u^k_{m,n} = R(\mathbf{h}; u^k_{m,n}), \tag{2}$$

that are powerful in the sense that their solutions $(u^k_{m,n})$ preserve significant properties of the solution of (1). (In (2) and further on, $\mathbf{h}$ represents the vector $(\Delta t, \Delta x, \Delta y)$). One way of achieving this is to consider nonstandard finite difference schemes introduced in the eighties by Mickens (see [6] and the references therein). Schemes were empirically developed using a collection of rules set by Mickens. In [2], two of the authors provided some mathematical justifications for the success of these empirical procedures. In particular, they unambiguously defined nonstandard finite difference schemes as follows by using two of Mickens' rules :

Definition 1. The scheme (2) is called a nonstandard finite difference method if at least one of the following conditions is met:

- In the first order discrete derivatives $\partial_{\Delta t} u^k_{m,n}, \partial_{\Delta x} u^k_{m,n}$ and $\partial_{\Delta y} u^k_{m,n}$ that occur in (2), the traditional denominators Δt, Δx and Δy are replaced by nonnegative functions $\phi_1(\Delta t)$, $\phi_2(\Delta x)$ and $\phi_3(\Delta y)$ such that, for $j = 1, 2, 3$,

$$\phi_j(z) = z + O(z^2) \quad \text{as } 0 < z \to 0. \tag{3}$$

- In the expression $R(\mathbf{h}; u^k_{m,n})$, nonlinear terms that occur in $r(u)$ are approximated in a nonlocal way, i.e. by a suitable function of several points of the mesh (see, e.g., (7_2) and (30)).

The power of the nonstandard finite difference method over the standard ones is expressed in the next definition also due to two of the authors [1,2].

Definition 2. Assume that the solutions of Eq. (1) satisfy some property $\mathscr{P}$. The numerical scheme (2) is called (qualitatively) stable with respect to property $\mathscr{P}$ (or $\mathscr{P}$-stable) if for every value of the step-sizes $\Delta t > 0$, $\Delta x > 0$ and $\Delta y > 0$ the set of solutions of (2) satisfies $\mathscr{P}$.

The paper is organized as follows. In the next section, we present exact finite difference schemes for (1) in the particular cases of the linear reaction $r(u) = \lambda u$ and of the logistic growth reaction $r(u) = \lambda u(1-u)$. The situation regarding general reactions $r(u)$ is addressed in two steps. Firstly, in Section 3, under the assumption that the partial differential equation in (1) has only hyperbolic fixed-points, we design two new schemes that are elementary stable and we provide some numerical examples. Secondly, in Section 4, we investigate two new stability properties as particular cases of elementary stability which, have the additional capability of describing qualitatively non-hyperbolic fixed points. Section 5, as an application of the results of Section 4 to a cubic reaction, presents a systematic way of deriving schemes by nonlocal approximation of nonlinear terms. The results are illustrated by numerical examples. The last section is devoted to concluding remarks where possible extensions are mentioned.

2. Exact finite difference schemes

We consider two particular reactions $r(u)$, which will serve as motivation for the design of non-standard schemes for (1) in the general case.

It is well-known that the evolution semigroup or the solution operator $t \to E(t)(\cdot)$ of (1) is

$$u(t,x,y) := E(t)f(x,y) = \begin{cases} f(x-at, y-bt)e^{\lambda t} & \text{if } r(u) = \lambda u, \\ \dfrac{f(x-at,y-bt)}{e^{-\lambda t} + (1-e^{-\lambda t})f(x-at,y-bt)} & \text{if } r(u) = \lambda u(1-u). \end{cases} \tag{4}$$

It is shown in [7] that, under the functional relation

$$\Delta x = a\Delta t \quad \text{and} \quad \Delta y = b\Delta t \tag{5}$$

between step sizes, the evaluation process

$$u_{m,n}^{k+1} = E(t_{k+1})f(m\Delta x, n\Delta y) \tag{6}$$

leads to the partial difference equations below, which are the so-called exact schemes [6] for (1):

$$\begin{cases} \dfrac{u_{m,n}^{k+1} - u_{m-1,n-1}^{k}}{(e^{\lambda\Delta t}-1)/\lambda} = \lambda u_{m-1,n-1}^{k} & \text{if } r(u) = \lambda u, \\ \dfrac{u_{m,n}^{k+1} - u_{m-1,n-1}^{k}}{(e^{\lambda\Delta t}-1)/\lambda} = \lambda u_{m-1,n-1}^{k}(1-u_{m,n}^{k+1}) & \text{if } r(u) = \lambda u(1-u). \end{cases} \tag{7}$$

3. Elementary stable schemes

On performing the change of variables

$$t \to t, \ x \to x + at, \ y \to y + bt; \quad U(t) := u(t, x+at, y+bt), \tag{8}$$

the advection–reaction equation (1) becomes, for a fixed $(x,y) \in \mathbb{R}^2$, an initial value problem for ordinary differential equations:

$$\frac{dU}{dt} = r(U), \quad U(0) = f(x,y). \tag{9}$$

This permits us to now extend the linear stability analysis of ordinary differential equations to the partial differential equation in (1). In fact, by a fixed-point or critical point of the said partial differential equation, we mean any zero $\tilde{u}$ of the function r: $r(\tilde{u}) = 0$. With $\tilde{u}$ a hyperbolic fixed-point, i.e., a fixed-point such that

$$J \equiv r'(\tilde{u}) \neq 0, \tag{10}$$

we associate the solution

$$\varepsilon(t,x,y) = \varepsilon^0(x-at, y-bt)e^{Jt} \tag{11}$$

of the linearized equation

$$\varepsilon_t + a\varepsilon_x + b\varepsilon_y = J\varepsilon, \quad \varepsilon(0,x,y) = \varepsilon^0(x,y). \tag{12}$$

Then, Hartman and Grobman's theorem (see, e.g., [11]) shows that the solution u of (1) in a neighborhood of $\tilde{u}$ and the solution ε of (12) in a neighborhood of 0 are such that the deviation $u-\tilde{u}$ and ε have both, along the line passing through $(0,x,y)$ and parallel to the vector $\langle 1,a,b\rangle$, the same asymptotic behaviour as $t\to\infty$. Thus, the next definition.

Definition 3. A hyperbolic fixed-point $\tilde{u}$ is called linearly stable provided that $\lim_{t\to\infty}\varepsilon(t,x+at,y+bt)=0$ or equivalently $J<0$ in (11). Otherwise, the fixed-point is called linearly unstable.

We assume henceforth that (5) holds. With the notation (2) in mind and $J_{\mathbf{h}}\varepsilon^k_{m,n}$ the linear term in $\varepsilon^k_{m,n}$ of the Taylor expansion around $\tilde{u}$ of $R(\mathbf{h};\tilde{u}+\varepsilon^k_{m,n})$, the discrete analogue of (12) is

$$\partial_{\Delta t}\varepsilon^k_{m,n}+a\partial_{\Delta x}\varepsilon^k_{m,n}+b\partial_{\Delta y}\varepsilon^k_{m,n}=J_{\mathbf{h}}\varepsilon^k_{m,n}. \tag{13}$$

Definition 4. Assume that a hyperbolic fixed-point $\tilde{u}$ of the partial differential equation in (1) is a solution of the difference scheme (2). We say that the constant solution or fixed-point $\tilde{u}$ is linearly stable or unstable according as $\varepsilon^k_{m+k,n+k}$ tends to 0 or not for $k\to\infty$, where $\varepsilon^k_{m,n}$ is a solution of the difference scheme (13) for any given small enough ε^0.

Definition 5. The finite difference scheme (2) is called elementary stable if for any value of the step size $\mathbf{h}$, its only fixed-points $\tilde{u}$ are those of the partial differential equation in (1), the linear stability properties of each $\tilde{u}$ being the same for both the partial differential equation and the discrete scheme.

Our main result in this section is the next theorem where two new elementary stable schemes are presented in the spirit of (7).

Theorem 6. *Let ϕ satisfying* (3) *be such that*

$$0<\phi(z)<1 \quad \textit{for } z>0. \tag{14}$$

Assume that the partial differential equation in (1) *has a nonzero finite number of fixed-points $\tilde{u}$, all being hyperbolic. Put $q=\max\{|r'(\tilde{u})|;r(\tilde{u})=0\}$. Then, subject to* (5), *the two nonstandard schemes*

$$\frac{u^{k+1}_{m,n}-u^k_{m-1,n-1}}{\phi(q\Delta t)/q}=\begin{cases} r(u^k_{m-1,n-1})\\ r(u^{k+1}_{m,n})\end{cases} \tag{15}$$

are elementary stable.

Proof. Let $\tilde{u}$ be a hyperbolic fixed-point of (1). The discrete error equation (13) becomes

$$\frac{\varepsilon^{k+1}_{m,n}-\varepsilon^k_{m-1,n-1}}{\phi(q\Delta t)/q}=\begin{cases} J\varepsilon^k_{m-1,n-1}\\ J\varepsilon^{k+1}_{m,n}\end{cases} \quad \text{and, thus,}\quad \varepsilon^k_{m+k,n+k}=\begin{cases} (1+\phi J/q)^k\varepsilon^0_{m,n}\\ (1-\phi J/q)^{-k}\varepsilon^0_{m,n}.\end{cases}$$

If $\tilde{u}$ is linearly stable for the partial differential equation, i.e., $J<0$, then by definition of q and the condition (14), we have

$$|1+\phi J/q|=1-\phi|J|/q<1 \quad \text{and} \quad |(1-\phi J/q)^{-1}|=(1+\phi|J|/q)^{-1}<1,$$

which imply that $\varepsilon^k_{m+k,n+k}$ tends to 0 as $k \to \infty$. This means that $\tilde{u}$ is linearly stable for the difference schemes. Analogously, if $\tilde{u}$ is linearly unstable for the partial differential equation, i.e., $J > 0$, then the linear instability of $\tilde{u}$ for the difference schemes follows from

$$|1 + \phi J/q| = 1 + \phi J/q > 1 \quad \text{and} \quad |(1 - \phi J/q)^{-1}| = (1 - \phi J/q)^{-1} > 1. \qquad \square$$

Remark 7. 1. The following standard finite difference schemes are elementary unstable:

$$\frac{u^{k+1}_{m,n} - u^k_{m,n}}{\Delta t} + a\,\frac{u^k_{m,n} - u^k_{m-1,n}}{\Delta x} + b\,\frac{u^{k+1}_{m-1,n} - u^k_{m-1,n-1}}{\Delta y} = \begin{cases} r(u^k_{m-1,n-1}) \\ r(u^{k+1}_{m,n}). \end{cases} \tag{16}$$

2. For the purpose of comparison with (16), the nonstandard schemes (15) may be written in the form

$$\frac{u^{k+1}_{m,n} - u^k_{m,n}}{\phi(q\Delta t)/q} + a\,\frac{u^k_{m,n} - u^k_{m-1,n}}{a\phi(q\Delta x/a)/q} + b\,\frac{u^k_{m-1,n} - u^k_{m-1,n-1}}{b\phi(q\Delta y/b)/q} = \begin{cases} r(u^k_{m-1,n-1}) \\ r(u^{k+1}_{m,n}). \end{cases} \tag{17}$$

Likewise, the left hand-sides of the exact schemes (7) could be written in the nonstandard form in (17) with appropriate functions ϕ.

3. If $r(u)$ is a constant with respect to u, (16) is the well-known Lax scheme (see, e.g., [4]). Thus, our methods (15) or (17) are nonstandard extensions of the Lax scheme.

4. The relation (5) plays an important role. It is a sufficient condition for stability in the sense of Lax-Richtmyer [10] of the Lax scheme (16) when $r(u)$ is constant in u. On the other hand, (5) is essential for stability of nonstandard schemes with respect to the important physical property of boundeness and positivity of solution (Definition 2). Indeed, if we assume that there holds for (1) the implication

$$0 \leqslant f(x,y) \leqslant 1 \Rightarrow 0 \leqslant u(t,x,y) \leqslant 1, \tag{18}$$

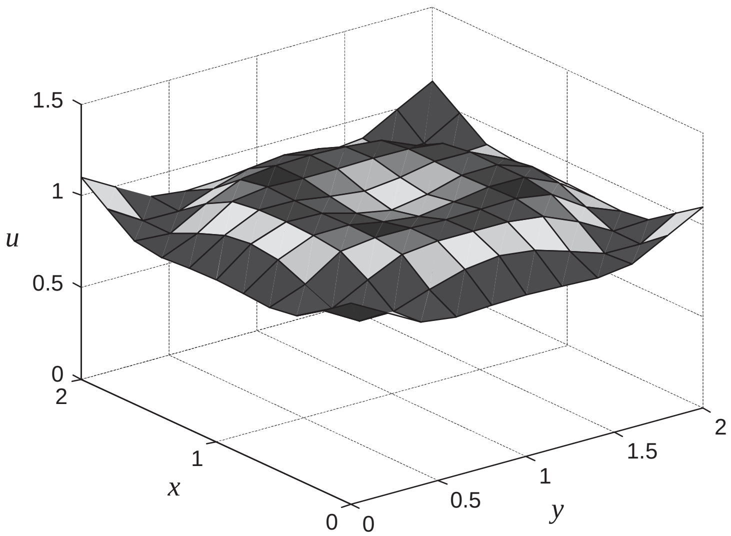

Fig. 1. Standard scheme (16_1).

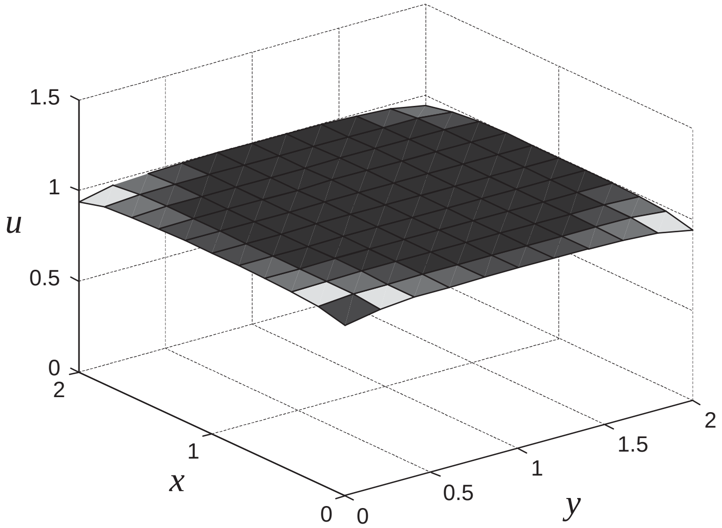

Fig. 2. Nonstandard scheme (15_1).

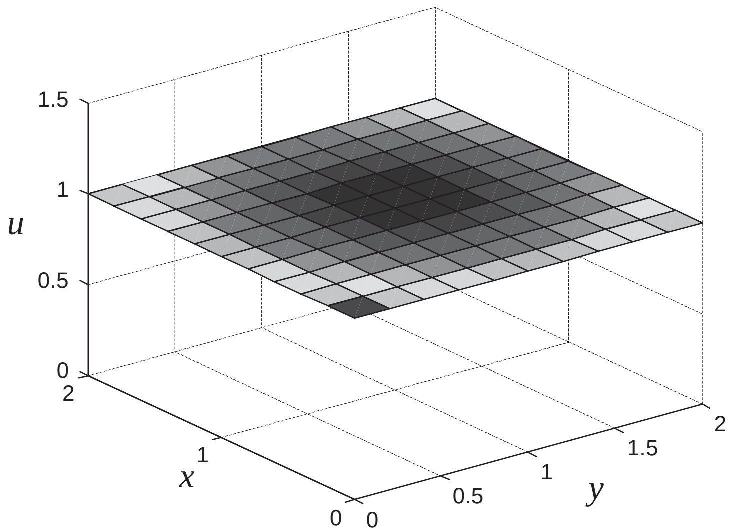

Fig. 3. Exact scheme (7_1).

then the schemes (15) are stable with respect to (18) (i.e., $0 \leqslant u^0_{m,n} \leqslant 1 \Rightarrow 0 \leqslant u^k_{m,n} \leqslant 1$) whenever (5) holds and the reaction satisfies the relation $-s \leqslant r(s)/q \leqslant 1-s$ for $0 \leqslant s \leqslant 1$. In particular, (18) is valid in the case of (4), with $\lambda \leqslant 0$ for (4_1), and the corresponding schemes (15) are stable with respect to (18).

As an illustration of the power of our nonstandard scheme (15_1) over the standard one (16_1), we have the self-explanatory Figs. 1–3. These figures concern the numerical solution at time $t = 1$ of

Eq. (1) with logistic reaction $r(u)=\lambda u(1-u)$ where $\lambda=10$, $f(x,y)=\mathrm{e}^{-x^2-y^2}$, $a=b=1$, $\Delta t=0.2$ and $\phi(\Delta t)=(1-\mathrm{e}^{-10\Delta t})/10$.

4. Stability with respect to monotonicity

To address the case of non-hyperbolic fixed points, it is natural, in view of the preceding two sections, to consider discrete schemes of the general form

$$u_{m,n}^{k+1}=F(h;u_{m-1,n-1}^{k}), \tag{19}$$

where the function F is assumed to be, in both arguments, as smooth as needed and the parameter h represents the time step size Δt, which throughout the whole section is related to Δx and Δy by (5). We also assume that the difference scheme (19) is consistent with the differential equation (1) (see [10]) such that, for any $u\in\mathbb{R}$ and $h>0$, we have

$$F(0;u)=u \quad \text{and} \quad \frac{\partial F}{\partial h}(0;u)=r(u). \tag{20}$$

Let us note that consistency implies that (20) is satisfied when u is the solution of (1).

We introduce two new properties of solutions of (1) and their discrete counterparts for (19). The first property is a consequence of the group property of the solution operator $E(t)(\cdot)$ and of the fact that we assumed that (1) has a unique solution. It follows indeed from this that

$$f(x,y)\leqslant g(x,y)\Rightarrow E(t)f(x,y)\leqslant E(t)g(x,y), \quad \forall t\geqslant 0. \tag{21}$$

Property (21) will be called the "monotone dependence of solutions of (1) with respect to initial values". Regarding the difference equation (19), we have the following result.

Theorem 8. *The condition*

$$\frac{\partial F}{\partial u}(h;u)\geqslant 0 \quad \forall u\in\mathbb{R} \text{ and } h>0 \tag{22}$$

is necessary and sufficient for the difference scheme (19) *to be stable with respect to monotone dependence on initial values, i.e.,*

$$f(x_m,y_n)\leqslant g(x_m,y_n)\Rightarrow E^k f(x_m,y_n)\leqslant E^k g(x_m,y_n) \quad \forall k\geqslant 0 \tag{23}$$

where $k\to E^k(\cdot)$ is the discrete solution operator associated with (19).

Proof. Set $u_{m,n}^k := E^k f(x_m,y_n)$ and $v_{m,n}^k := E^k g(x_m,y_n)$. The condition $(\partial F/\partial u)(h;u)\geqslant 0$ means that the function $u\to F(h,u)$ is increasing, which shows that $u_{m,n}^k\leqslant v_{m,n}^k$ whenever $f(x_m,y_n)\leqslant g(x_m,y_n)$. Conversely, if we assume that the condition (22) is not satisfied, we may find $\tilde{h}$, $\tilde{u}\in\mathbb{R}$ and $\varepsilon>0$ such that $(\partial F/\partial u)(\tilde{h};u)<0$ for any $u\in(\tilde{u}-\varepsilon,\tilde{u}+\varepsilon)$. For the difference scheme (19) initiated at $f=\tilde{u}-\varepsilon$ and $g=\tilde{u}+\varepsilon$, we would have the contradiction $v_{m,n}^1-u_{m,n}^1=F(\tilde{h};g)-F(\tilde{h};f)=2\varepsilon(\partial F/\partial u)(\tilde{h};\xi)<0$. □

The second property of monotonicity to be considered is described as follows. Due to (8) and to the autonomous nature of (9), every solution $u(t,x+at,y+bt)$ of (1), along the line passing through $(0,x,y)$ and parallel to the vector $\langle 1,a,b\rangle$, is either increasing or decreasing in t on the whole interval $[0,\infty)$. The increasing and decreasing solutions are separated by fixed points $\tilde{u}$ of the partial differential equation in (1). For the discrete scheme (19), we adopt the following definition.

Definition 9. Under the condition (5), the difference scheme (19) is stable with respect to the property of monotonicity of solutions if, for every initial data f and fixed (x,y), the solution $u^k_{m+k,n+k}$ of (19) is an increasing or a decreasing sequence in k according as the solution $u(t,x+at,y+bt)$ of Eq. (1) is increasing or decreasing in t.

Theorem 10. *Assume that the difference scheme* (19) *is stable with respect to monotone dependence on initial values. Assume also that, for every* $h>0$, *the equations*

$$u=F(h,u) \quad \textit{and} \quad r(u)=0 \tag{24}$$

in u *have the same roots considered with their multiplicities. Then the difference scheme* (19) *is stable with respect to monotonicity of solutions.*

Proof. We consider the solution of the differential equation (1) and of the difference scheme (19) both initiated at a given f. We fix $(x,y)\in\mathbb{R}^2$ and set $u^0:=f(x,y)$. The situation being easy when u^0 is fixed point or a root of the equations in (24), we assume first that $r(u^0)>0$ and denote by $\tilde{u}$ the smallest fixed point of (1) and (19), which is greater than u^0. If there are no fixed points greater than u^0, we set $\tilde{u}=\infty$. Then for every $u\in[u^0,\tilde{u})$ we have $r(u)>0$. For the considered value of u^0, the solution $u(t,x+at,y+bt)$ of (1) is an increasing function of $t\in[0,\infty)$ and $u(t,x+at,y+bt)\in[u^0,\tilde{u})$ for $t\in[0,\infty)$. We will show that the solution $(u^k_{m+k,n+k})$ of (19) is an increasing sequence in k. First we would like to prove that

$$F(h,u)>u \quad \text{for } u\in[u^0,\tilde{u}) \text{ and } h>0. \tag{25}$$

Assume the opposite, i.e., there exist $\hat{u}\in[u^0,\tilde{u})$ and $\bar{h}$ such that

$$F(\bar{h},\hat{u})<\hat{u}. \tag{26}$$

From $(\partial F/\partial h)(0;\hat{u})=r(\hat{u})>0$ that is given by (20), it follows that $F(h,\hat{u})-F(0,\hat{u})>0$ for small enough h. Thus, again (20) yields

$$F(h;\hat{u})>\hat{u} \text{ for small enough } h. \tag{27}$$

It follows from (26) and (27) that there exists $\hat{h}\in(0,\bar{h})$ such that $F(\hat{h};\hat{u})=\hat{u}$. Thus $\hat{u}$ is a fixed point of $u\to F(h,u)$ for $h=\hat{h}$, which is a contradiction since this function has no fixed points in $(u^0,\tilde{u})$.

Consider now the solution $(u^k_{m+k,n+k})$ of (19). If $\tilde{u}$ is a fixed point, then the stability with respect to monotone dependence on initial values implies that $u^k_{m+k,n+k}\leqslant\tilde{u}$, $k=1,2,\ldots$. (If $\tilde{u}=\infty$ the inequality $u^k_{m+k,n+k}<\tilde{u}$ is obvious.) Using (25) and (19), it is easy to see inductively that $u^k_{m+k,n+k}\in(u^0,\tilde{u}]$ and $u^k_{m+k,n+k}\geqslant u^{k-1}_{m+k-1,n+k-1}$. In a similar way, one proves that the solution $(u^k_{m+k,n+k})$ of (19) initiated at u^0 is decreasing when $r(u^0)<0$. □

One shortcoming of the concept of elementary stability introduced in the preceding section is that it fails to describe the behavior of solutions around non-hyperbolic fixed points. The next result permits to fill this gap.

Theorem 11. *Under the assumptions of Theorem* 10, *the difference scheme* (19) *is elementary stable.*

Proof. Condition (24) implies that for every $h>0$, the difference scheme (19) has the same fixed points and the same hyperbolic fixed points as Eq. (1). Let $\tilde{u}$ be a fixed point of (1), which is linearly stable, i.e. $\tilde{u}$ is a hyperbolic fixed point of (1) such that $r'(\tilde{u})<0$. Therefore $\tilde{u}$ is a simple root of $F(h,u)=u$. The analogue of (13) for (19) being $\varepsilon_{m,n}^{k+1}=(\partial F/\partial u)(h,\tilde{u})\varepsilon_{m-1,n-1}^{k}$, we will show that $0\leqslant(\partial F/\partial u)(h,\tilde{u})<1$ for $h>0$, which implies (Definition 4) that $\tilde{u}$ is a linearly stable fixed point of (19). Since $r'(\tilde{u})<0$, for $\Delta u>0$ small enough, we have $r(\tilde{u}-\Delta u)>r(\tilde{u})=0$. It follows from (25) that $F(h,\tilde{u}-\Delta u)>\tilde{u}-\Delta u$. In the same way, using an analogue of (25), $F(h,\tilde{u}+\Delta u)<\tilde{u}+\Delta u$ for Δu small enough. Therefore (22) yields

$$0\leqslant\frac{F(h,\tilde{u}+\Delta u)-F(h,\tilde{u}-\Delta u)}{2\Delta u}<\frac{\tilde{u}+\Delta u-\tilde{u}+\Delta u}{2\Delta u}=1.$$

Passing to the limit when $\Delta u\to 0$, we obtain $0\leqslant(\partial F/\partial u)(h,\tilde{u})<1$ because $\tilde{u}$ is a simple zero of $F(h,u)=u$. Thus, $\tilde{u}$ is a linearly stable fixed point of (19).

If $\tilde{u}$ is a fixed point of (1) which is linearly unstable, i.e., $r'(\tilde{u})>0$, it can be shown in a similar way that $(\partial F/\partial u)(h,\tilde{u})>1$, which implies that $\tilde{u}$ is a linearly unstable fixed point of (19). □

5. A cubic reaction term

We consider the advection–reaction equation (1) with the cubic reaction term

$$r(u)=\lambda u^2(1-u),\quad \lambda>0, \tag{28}$$

that occurs in elementary model for combustion (see [8,9]). The interest of (1) and (28) hinges on the fact that its fixed point $\tilde{u}_1=0$ is non-hyperbolic, whereas $\tilde{u}_2=1$ is a linearly stable hyperbolic fixed-point. However, under the condition (5) and for fixed (x,y), the solution $u(t,x+at,y+bt)$, with initial value $u^0:=f(x,y)$, has interesting monotonic properties summarized in the following table:

Initial condition	Monotonicity	Limit as $t\to\infty$
$u^0\in(-\infty,0)$	Increasing	0
$u^0\in(0,1)$	Increasing	1
$u^0\in(1,+\infty)$	Decreasing	1

(29)

This shows that the non-hyperbolic fixed point $\tilde{u}_1=0$ attracts the solutions below it and repulses the solutions above it. We shall apply the theory of the previous section to the design of nonstandard schemes, for (1) and (28), which produce numerical solutions with the same properties. To this end, with real parameters α and β, we consider the family of schemes

$$\begin{aligned}\frac{u_{m,n}^{k+1}-u_{m-1,n-1}^{k}}{\lambda\phi(h)}&=\alpha(u_{m-1,n-1}^{k})^2+(1-\alpha)u_{m-1,n-1}^{k}u_{m-1,n-1}^{k+1}\\&\quad-\beta(u_{m-1,n-1}^{k})^3-(1-\beta)(u_{m-1,n-1}^{k})^2u_{m-1,n-1}^{k+1}\end{aligned} \tag{30}$$

or equivalently

$$u_{m,n}^{k+1} = F(h; u_{m-1,n-1}^{k}) \text{ with } F(h;u) = \frac{u + \lambda\phi(h)\alpha u^2 - \lambda\phi(h)\beta u^3}{1 + \lambda\phi(h)(\alpha-1)u + \lambda\phi(h)(1-\beta)u^2}. \tag{31}$$

We seek a set of values of the parameters α and β for which Theorems 8, 10 and 11 apply. Condition (22) simplifies to

$$((\beta^2-\beta)u^2 - 2\beta(\alpha-1)u + \alpha^2 - \alpha)u^2\lambda^2\phi^2 - (2\beta+1)\lambda\phi \times \left(u - \frac{2}{2\beta+1}\right)^2 + \frac{\alpha^2\lambda\phi}{2\beta+1} + 1 \geqslant 0. \tag{32}$$

Simple manipulations show that (22) or (32) is met if

$$\alpha \geqslant 1, \quad \beta < -1/2, \tag{33}$$

and the function ϕ satisfying (3) is such that

$$0 < \phi < c \quad \text{where } c = -(2\beta+1)/(\lambda\alpha^2). \tag{34}$$

A possible choice for the function ϕ is $\phi(h) = c(1 - \mathrm{e}^{-h/c})$. Furthermore, the function $F(h,u)$ can be written in the form

$$F(h;u) = u + r(u)\,\frac{\lambda\phi}{1 + \lambda\phi(\alpha-1)u + \lambda\phi(1-\beta)u^2},$$

which under (33) yields $F(h,u) = u \Leftrightarrow r(u) = 0$ for every $h > 0$. Thus, under conditions (33) and (34), Theorems 8, 10 and 11 imply elementary stability of the scheme (31) as well as its stability with respect to monotone dependence on initial values and monotonicity of solutions.

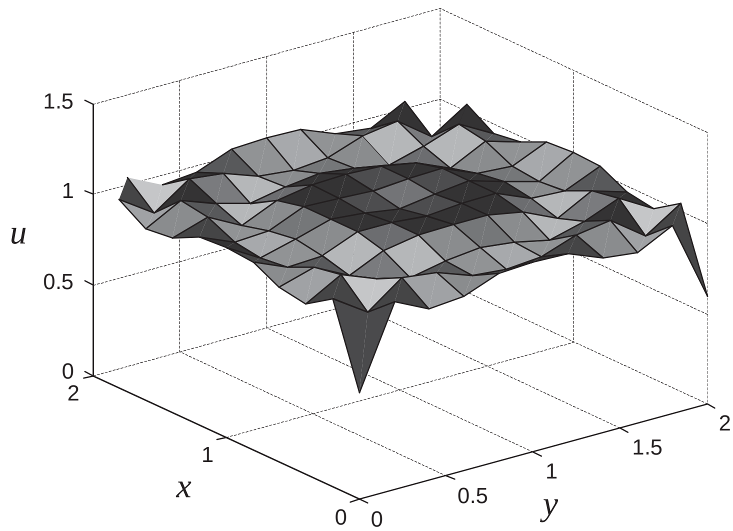

Fig. 4. Scheme (30) with $\alpha = \beta = 1$.

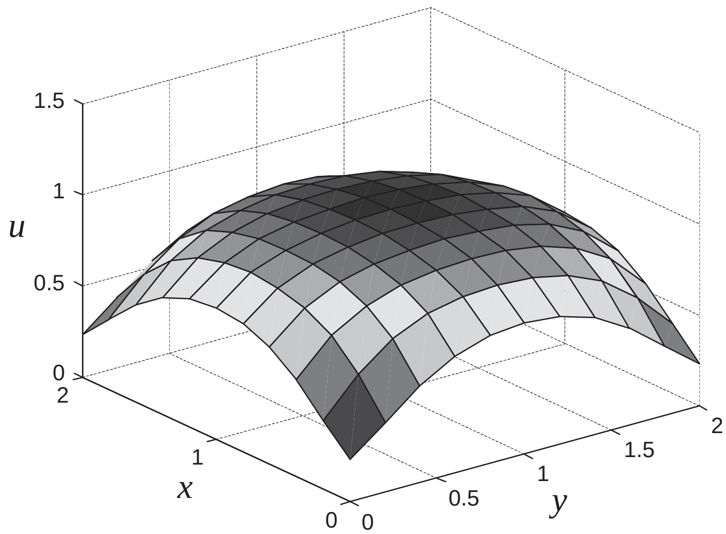

Fig. 5. Scheme ([30]) with $\alpha = -\beta = 1$.

At the end, we have Figs. 4 and 5 below regarding numerical solutions of (1) and (28) at the time $t = 1$ by using the standard ($\alpha = \beta = 1$) and nonstandard ($\alpha = 1,\ \beta = -1$) schemes (30) for $a = b = 1,\ \lambda = 10,\ \Delta t = 0.2$ with initial condition $f(x, y) = \mathrm{e}^{-x^2-y^2}$. These figures illustrate whether or not the discrete solutions remain in the interval [0,1]. Furthermore, the standard scheme is not stable with respect to monotone dependence on initial values since the discrete solution intersects with the constant solution 1.

Remark 12. The perturbation procedure used in the construction of the family of discrete schemes (30) works also for other partial differential equations and in the case of hyperbolic fixed points. This is done in [3] for the reaction–diffusion equation and for the logistic growth equation.

6. Concluding remarks

The advection terms of (1) often occur in more complex equations, in which other terms, need numerical approximations. These include among others: the advection–diffusion–reaction equations [5], the transport equations and the symmetric Friedrichs systems [4]. Classically, it is from the schemes obtained for the advection equation that one generates general schemes for the more complex equations. The complex terms under consideration in this paper are reactions. The starting point is the exact schemes provided in [7] for the advection equation with the logistic growth reaction or without any reaction. (Exact schemes for some other particular reactions are given in [5].) Thereafter, we have introduced some nonstandard extensions of the Lax scheme for the advection equation with general reaction terms. In the case of hyperbolic fixed-points, our construction, by renormalization of the denominators of discrete derivatives, extends to partial differential equations the results in [6,2].

A further more general approach, based on nonlocal approximation of nonlinear terms and valid for both hyperbolic and nonhyperbolic fixed-points, is implemented. The power of our schemes over the standard ones is that they are reliable numerical simulations that preserve the linear stability and monotonicity properties of the exact solutions.

Our interest for future research is to extend this study to more general equations such as those mentioned above which, include further complex terms apart from the advection and the reaction terms. In this regard, a nonstandard approach to advection–reaction–diffusion equations is presented in [5].

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